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RATIONAL EXPECTATIONS BUSINESS CYCLES
IN SEARCH EQUILIBRIUM

Peter Diamond
Drew Fudenberg

No. 465 October 1987

massachusetts
institute of
technology

50 memorial drive
cambridge, mass. 02139

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If the profitability of production increases with the level of economic activity, then steady-state rational expectations equilibria need not be efficient. Diamond (1982) illustrated this point in a model of search equilibrium in which the rate of meeting trade partners increases with the number of potential partners. The presence of multiple steady state equilibria in that model suggests that there may be multiple rational expectations equilibrium paths for some initial positions. This paper provides such an example. For some initial positions, the belief that others will undertake many production possibilities makes it privately worthwhile to undertake many opportunities. This optimistic equilibrium path Pareto-dominates the "pessimistic" path on which agents correctly believe that others will only undertake a few production possibilities. There can also be equilibria in which traders correctly believe that the economy will alternative between optimistic and pessimistic phases. In these "endogenous business cycles," production waxes and wanes in an otherwise stationary environment. While these cycles are deterministic, they have more in common with "sunspot equilibria" than with the Pareto-efficient deterministic cycles occuring in optimal growth models , as they are based on the market imperfections implicit in costly search.

Because expectations can so strongly influence the economy, traders would prefer some way of coordinating on a "good" equilibrium. It is not unheard-of for governments to issue optimistic forecasts in the hope of inducing optimistic private forecasts; such forecasts could be self-fulfilling. The government may also directly intervene to stimulate demand and launch the economy on an optimistic path. We have not

attempted to model such policies directly, which would require an explicit treatment of the revision of expectations. However we do intend our model to be suggestive of the possible expectational role of aggregate demand management.

In Section 1 we briefly review the simple model presented previously. In Section 2 we consider the dynamics of the willingness to produce and consider two equilibrium paths converging to steady states. Section 3 shows that our model exhibits the Hopf bifurcation, so that equilibrium cycles can exist for some parameter values. Section 4 contains explicit calculation of some cycles.

1. Basic Model

All individuals are assumed to be alike. Instantaneous utility satisfies

$$U = u - c \tag{1}$$

where u is the utility from consumption of y units of output and c is the cost of production (disutility of labor). Lifetime utility is the present discounted value of instantaneous utility

$$V = \sum_{i=1}^{\infty} e^{-rt_i} U(t_i) \tag{2}$$

where t_i are the dates at which consumption and production occur. Individuals are assumed to maximize the expected value of lifetime utility.

Production opportunities arrive as a Poisson process. With arrival rate a, each individual learns of production opportunities. Each opportunity has y units of output and costs c ($c > 0$) units to produce. We assume that y is the same for all projects and c is an independent draw

from the distribution $G(c)$ with support on $(\underline{c},+\overline{c})$, where $\overline{c} > \underline{c} > 0$. There are two further restrictions. We assume that individuals cannot consume the products of their investment, but trade their own output for that produced by others. This represents the small extent to which individuals consume their own output in a modern specialized economy. We also assume that individuals cannot undertake a production project if they have unsold produced output on hand. The fact that all trades involve individuals with y units to sell implies that all units are swapped on a one-for-one basis, and promptly consumed. Thus individuals have 0 or y units for sale. The former are looking for production opportunities and are referred to as unemployed. The latter are trying to sell their output and are referred to as employed.

The trading process is such that for each individual the arrival of potential trading partners is a Poisson process with arrival rate $b(e)$, $b' > 0$, $b'' \leq 0$, and $b(0) = 0$, where e is the fraction of the population employed, which equals the stock of potential trading partners. This lag in the trading process is intended to represent the time needed to sell goods; when there are fewer potential customers, sales are less frequent. The average time that consumer goods spend in inventories is assumed to increase as the rate of sales declines. It is assumed that there is no money and no credit market, so those with nothing to sell are unable to buy. The economy is assumed to be sufficiently large that the expected values of potential production and trade opportunities are realized. The employment rate falls from each completed transaction, as a previously employed person becomes eligible to undertake a production opportunity, and rises whenever a production opportunity is undertaken.

If all opportunities are undertaken that cost less than c^*, the time derivative of the level of inventories satisfies

$$\dot{e} = -eb(e) + a(1-e)G(c^*) \tag{3}$$

That is, each of the e employed (per capita) faces the probability $b(e)$ of having a successful trade meeting and being freed to seek a new opportunity. Each of the $1-e$ unemployed (per capita) has the flow probability a of learning of an opportunity, of which the fraction $G(c^*)$ is undertaken.

2. Multiple Equilibria

The only decisions made by the agents in our model are which production opportunities to undertake. Their choices will depend on how they expect the economy to evolve, because inventory is more valuable when there are more potential trading partners. We make the "rational expectations" (or "perfect foresight") assumption that all agents correctly anticipate the economy's future trajectory.

Let us denote the optimized expected present discounted value of lifetime utility for employed and unemployed by W_e and W_u respectively. Along a fixed equilibrium trajectory W_e and W_u are functions of e. Each of these values satisfies the condition that its level times the utility discount rate equals the expected value of the flow of instantaneous utility plus the expected capital gains from a change in status and from a change in the employment rate:

$$rW_e = b[u - W_e + W_u] + W_e'\dot{e} \tag{4a}$$

$$rW_u = a\int_0^{c^*} [W_e - W_u - c]dG + W_u'\dot{e} \tag{4b}$$

With probability b, an employed person has a trade opportunity giving rise to instantaneous utility u and a change in status to unemployed. Each unemployed person accepting a production opportunity has an instantaneous utility -c and a change in status to employed. Since individuals will accept any project that raises expected utility, the unemployed accept any project costing less than $c^* \equiv W_e - W_u$.

Taking the difference between the equations in (4), and noting that $(W_e'-W_u')\dot{e}$ is $\dot{c}^*$ we have the equation

$$rc^* = b(u-c^*) - a\int_0^{c^*}(c^*-c)dG + \dot{c}^* \qquad (5)$$

Equation (5) gives a necessary condition for the optimal willingness to produce along a path. For optimality we also have a transversality condition, that c^* be uniformly bounded in t, and that it not reach zero when e (and so b(e)) is positive. That is, beliefs about willingness to invest must be asymptotically correct as well as instantaneously justifiable. Recognizing that b is a function of e and that the $\dot{e}$ is determined by (3) we have a system of two differential equations in two variables, e and c^*. A solution path to these two equations is a rational expectations equilibrium if e satisfies the initial condition and c^* satisfies a suitable transversality condition.

To analyze these equations in a phase diagram, we begin with the locus $\dot{c}^* = 0$. This corresponds to the willingness to invest of someone with naive expectations. From (5) we have the equation for the locus $\dot{c}^* = 0$. The locus passes through the origin. Differentiating implicitly we have

$$\left.\frac{dc^*}{de}\right|_{\dot{c}^*=0} = \frac{(u-c^*)b'(e)}{r+b(e)+aG(c^*)} > 0$$

$$\left.\frac{d^2c^*}{de^2}\right|_{\dot{c}^*=0} = \frac{(u-c^*)b''(e)-2b'(e)\,\frac{dc^*}{de} - aG'(\frac{dc^*}{de})^2}{r+b(e)+aG(c^*)} < 0 \tag{6}$$

We note that the locus $\dot{c}^* = 0$ is increasing and concave in e, and bounded above by u. We show it in Figure 1 along with the locus $\dot{e} = 0$ and the directions of motion. The curve $\dot{e} = 0$ rises along the vertical axis to $\underline{c}$, increases up to the maximum sustainable inventory level and is again vertical above $\overline{c}$. For convenience, Figure 1 is drawn so that there are precisely three stationary points, at (0,0), (e_1,c_1^*), and (e_2,c_2^*) with $c_2^* < \overline{c}$. (In the next section, we present an example in which this is so.)

In Figure 2 we have added the trajectories going to (0,0) and (e_2,c_2^*). It is necessarily the case that the trajectory to (0,0) lies below $\dot{e} = 0$ at e_1, while the trajectory to (e_2,c_2^*) lies above $\dot{e} = 0$ at this point. It is natural to call the trajectory going to (0,0) the pessimistic path, and the one going to (e_2,c_2^*) the optimistic path. Both paths are rational expectations equilibria. Since they lie on either side of $\dot{e} = 0$ at e_1, there is necessarily a nondegenerate interval of initial conditions for which both paths are rational expectations equilibria.

Since trading opportunities are better the higher the employment rate, the optimistic path Pareto-dominates the pessimistic one. That is, if everyone is optimistic (i.e., believes the economy to be converging to e_2) the economy will converge to e_2 and trading opportunities will be good. However, if everyone is pessimistic (i.e., believes the economy to be converging to 0), the economy will converge to 0 and

trading opportunities will be poor. All agents would be better off if they could coordinate their expectations on the "good" equilibrium.

3. The Hopf Bifurcation and the Existence of Business Cycles

The fact that for some initial positions both optimistic and pessimistic beliefs are rational expectations equilibria suggests that there may also be equilibria with "endogenous business cycles," in which traders correctly believe that the economy will alternate between expanding and contracting phases.

We explore the possibility of cycles in this section by more thoroughly analyzing the dynamics of the system for the special case in which $b(e) = e$ and production costs are uniformly distributed on $[\underline{c}, \underline{c} + 1]$. Examining the system's behavior near its steady-states reveals that it can exhibit the Hopf bifurcation, so that cycles do indeed occur for some parameter values.

Consider then the following specialization of the equations for $\dot{e}$ and $\dot{c}^*$.

$$\dot{e} = \begin{cases} -e^2 & c^* \leq \underline{c} \\ -e^2 + a(1-e)(c^*-\underline{c}) & \underline{c} \leq c^* \leq \underline{c} + 1 \\ -e^2 + a(1-e) & c^* \geq \underline{c} + 1 \end{cases}$$

$$\dot{c}^* = \begin{cases} (r+e)c^* - ue & c^* \leq \underline{c} \\ (r+e)c^* - ue + a(c^*-\underline{c})^2/2 & \underline{c} \leq c^* \leq \underline{c} + 1 \\ (r+e)c^* - ue - a/2 + a(c^*-\underline{c}) & c^* \geq \underline{c} + 1 \end{cases} \qquad (7)$$

The loci along which $\dot{e}$ and c^* equal zero and the saddle point paths are depicted in Figures 1 and 2 for the parameters $a = 1$, $r = .1$, $\underline{c} = .5$, $u = 1.2$. We restrict attention to parameter values with $u < \underline{c} + 1$.

Thus intersections of the two stationary loci (other than at (0.0)) occur on the middle section of the two curves in (7).

In accordance with equation (6), the $\dot{c}^* = 0$ locus is concave; moreover in this special case the $\dot{e} = 0$ locus is convex for $\underline{c} < c^* < \bar{c}$. Thus there are either one, two, or three steady-states. Increasing r, holding the other parameters constant, the $\dot{c}^* = 0$ locus shifts down monotonically while the $\dot{e} = 0$ locus does not move. If the interest rate r is too high, the only steady-state is the trivial one (0,0). As agents become more patient, the $\dot{c}^* = 0$ locus rises, until for some $\bar{r} > 0$, there is exactly one steady-state in which production occurs. For lower interest rates there are two steady states with positive production and inventory levels, $e_1(r) < e_2(r)$. As r approaches zero, $e_1(r)$ decreases monotonically to zero, while e_2 increases monotonically to a level $\bar{e} < 1$. We can invert these relations. With other parameters fixed, for every e strictly between 0 and $\bar{e}$ there is a unique value of r, r(e), which is consistent with a steady state at e. Direct computation shows that

$$r(e) = [e/(1-e)] \left[\frac{a(u-\underline{c})(1-e)^2 + e^3/2 - e^2}{e^2 + a\underline{c}\,(1-e)}\right] \tag{8}$$

This curve is depicted in Figure 3 for the parameter values $a = 1$, $\underline{c} = .5$, $u = 1.2$.

From the relative slopes of the stationary curves, the phase diagram, Figure 1 reveals that (e_2, c_2^*) is a saddle point, while (e_1, c_1^*) may be either a spiral or a node. Moreover, (e_1, c_1^*) can be stable or unstable (i.e., a sink or a source). The saddle point paths converging to (e_2, c_2^*) and (0,0) are the optimistic and pessimistic equilibria discussed in the previous section. Any paths converging to (e_1, c_1^*) are

additional equilibria. When the paths spiral in, we have a continuum of equilibrium paths for initial values near e_1. The same result holds when we have a spiral out converging to a limit cycle, with the stationary cycle being another equilibrium.

If, as the interest rate varies, the paths through e_1 vary continuously from spiral in to spirals out, then one might expect that at intermediate values of r there would be paths which are closed cycles. For example, in purely linear systems, all paths are cycles at the "bifurcation point" where the spirals switch direction. This observation is extended to non-linear second-order systems by the Hopf Bifurcation Theorem [see for example Chow-Hale [1982]). Using Hopf, we will examine parameters for which cycles occur in the neighborhood of e_1. (The phase diagram shows that there cannot be cycles centered at e_2, for if $c^* > c_2^*$ and $e < e_2$ then c^* would grow without limit.) Cycles should be expected near steady-states that are spirals, such as e_1, and not those which are saddle points, like e_2.

To determine the system's behavior near e_1, we linearize the system (7) around (e_1, c_1^*). Setting $c_1 = \underline{c} + e_1^2/a(1-e_1)$, the resulting system is:

$$\begin{bmatrix} \dot{e} \\ \dot{c}^* \end{bmatrix} = \begin{bmatrix} -2e - e^2/(1-e) & a(1-e) \\ \underline{c} - u + e^2/a(1-e) & r+e+e^2/(1-e) \end{bmatrix} \begin{bmatrix} e - e_1 \\ c^* - c_1^* \end{bmatrix} \tag{9}$$

The trace of this matrix is

$$tr = r - e \; ; \tag{10}$$

its determinant is

$$d = -\left[[r+e+e^2/(1-e)]\cdot[2e+e^2/(1-e)] + [a(1-e)(\underline{c}-u+e^2/a(1-e))]\right] \tag{11}$$

and its eigenvalues are proportional to $tr \pm ((tr)^2-4d)^{1/2}$. (See for

example Coddington and Levinson [1955]). Equations (10) and (11) also hold for linearization around (e_2, c_2^*).

To see the types of equilibria we can calculate the trace and determinant for each point with $r > 0$ on the $r(e)$ curve in Figure 3. When the determinant is negative the equlilibrium is a saddle point. When the determinant is positive, it is a node or spiral depending on the sign of $(tr)^2 - 4d$. The node or spiral is stable when the trace is negative. In Figures* 4a, b, and c we show the locus of trace-determinant pairs for equilibria on the $r(e)$ curve. The three figures show the three possible patterns of first crossings of the horizontal axis of this locus. At $r = e = 0$, the trace is zero and the determinant is positive. This is the start of the locus on the right. At $r = 0$, $e = \overline{e}$, the trace is negative and the determinant is also negative. This is the end of the locus on the left. For the parameters resulting in the case shown in Figure 4a, there is no solution to $r = e$ with $e > 0$. In the other two cases, there is necessarily such a solution, although it may occur at a saddle point (Figure 4c) rather than at a spiral (Figure 4b).

To distinguish between the case in Figure 4a and the other two, we need to calculate dr/de in (8) evaluated at $e = 0$. When this derivative is greater than 1, we must have a crossing of the horizontal axis. Straightforward calculation shows that

$$\text{sign} \left.\frac{dr}{de}\right| = \text{sign}\ (u-2\underline{c}) \qquad (12)$$

Thus, provided $u > 2\underline{c}$, there is a solution to $r(e) = e$. Because of the continuity of the locus in Figure 4 to the parameters, we know that

*The Figures are drawn for parameter values a=1, r=.1, c=.5 and u=1.0, 1.2, 1.4.

there are such solutions along the e_1 section of $r(e)$ (i.e., with $d > 0$) for $u - 2\underline{c}$ sufficiently small, with a held constant.

Thus we can find parameters such that $r(e_1) = e_1$, so that the trace of (9) is zero, and such that $d > 0$, so the eigenvalues at e_1 are purely imaginary. For r just less than this value of e_1 the trace is positive, so the paths spiral out, while for r just above this value of e_1 the paths spiral in. This (plus "smoothness") is sufficient for the existence of cycles. This method of proff shows only that there is a range of values of a, $\underline{c}$, and u, such that there are cycles for a certain choice of r. The Hopf argument does not prove that cycles occur for an open set of parameters; doing so would require more detailed analysis. Also, it does not give us an idea of how large the cycles can be.

4. Calculated Examples of Cycles

Now we explicitly construct "large" business cycles in the case where $b(e) = e$ and all projects cost c, i.e., the distribution of costs is degenerate. In this case the business cycle has a simple form: all opportunities are accepted in "booms" as inventories grow from $\underline{e}$ to $\overline{e}$, and all are refused in "slumps" as inventories fall from $\overline{e}$ to $\underline{e}$.

The equations for $\dot{e}$ and $\dot{c}^*$ are now

$$\begin{array}{ll} \dot{e} = -e^2 & c^* < c \\ -e^2 < \dot{e} < -e^2 + a(1-e) & c^* = c \\ \dot{e} = -e^2 + a(1-e) & c^* > c \\ & \\ \dot{c}^* = (r+e)c^* - ue & c^* \leq c \\ \dot{c}^* = (r+e)c^* + a(c^*-c) - ue & c^* > c \end{array} \tag{13}$$

Solving numerically (as discussed in the Appendix), we have found many parameter values which yield cycles. Some of these are shown in Table 1. These examples all set $u = b = 1$, then specify a value of a (either .1, 1, or 10), and values for $\underline{e}$ and $\bar{e}$. Our program then solves for values of r and c which make $(\underline{e},\bar{e})$ into an equilibrium cycle. Because small changes in $\underline{e}$ lead to only small changes in r and c, our results suggest that cycles exist for a "large" set of parameters. Also shown in Table 1 is the value of e_2, the level of inventories in the optimistic steady state equilibrium.

TABLE 1

$u=b=1$

a	r	c	$\underline{e}$	$\bar{e}$	e_2
.1	.02	.562	.008	.045	.270
.1	.107	.512	.090	.135	.270
1.	.052	.567	.034	.103	.618
1.	.057	.574	.052	.103	.618
10.	.063	.582	.025	.153	.916

5. Conclusion

This paper has explored some of the possibilities inherent in a model of rational expectations equilibrium with trading externalities. The critical role played by expectations in the generation of multiple equilibria underlines the importance of a fuller understanding of how expectations are actually formed, and shows that the assumption of rational expectations on its own need not lead to precise predictions about how the economy will behave .

In a more positive light, we have shown how to construct rational expectations cycles on the basis of trading externalities and expectations. We do not think that actual cycles are triggered by widely

perceived floors and ceilings that have no role other than expectation coordination. Nor do we believe that cycles are merely coordinated by the mathematically similar and somewhat more realistic sounding sunspots (e.g., leading indicators being good predictors because they are believed to be good predictors). Rather, we feel that these simple models which illustrate expectational feedbacks need to be combined with more complex models of the economy which incorporate additional interactions including multiplier and accelerator effects.

REFERENCES

P. Diamond, "Aggregate Demand Management in Search Equilibrium," Journal of Political Economy, 90 (1982), pp. 881-894.

Chow, S. N. and J. K. Hale, Methods of Bifurcation Theory, Springer-Verlag, 1982.

Coddington, E. A. and N. Levinson, Theory of Ordinary Differential Equations, McGraw Hill, NY 1955.

APPENDIX

We describe the calculation of the cycles shown in Table 1. The $\dot{c}^* = 0$ locus is concave as before. The locus $\dot{e} = 0$ is the vertical segments $e = 0$, for $c^* < c$, and $e^2/(1-e) = a$ for $c^* > c$, joined by the horizontal segment $c^* = c$. We consider values for r low enough that the curves intersect three times: at the origin, on the horizontal segment, and on the vertical segment. At $e = e_2$, c^* exceeds c, so all opportunities are accepted. At $e = 0$, c^* is less than c and no opportunities are taken. At $e = e_1$, $c^* = c$, so agents are indifferent between accepting and rejecting projects, and we can specify the fraction which accept so that $\dot{e} = 0$. We now build a cycle along which employment oscillates between $\underline{e}$ and $\overline{e}$, with $0 < \underline{e} < e_1$ and $e_1 < \overline{e} < e_2$. At the the switchpoints $\underline{e}$ and $\overline{e}$, we must have $c^* = \underline{c}$.

Let c^*_b and c^*_s be the reservation value in booms and slumps, respectively. They must follow the differential equation (13), which is easily solved given a time path of e. The initial conditions are provided by the requirement that $c^*(t) = c$ for each t at which $e(t) = \underline{e}$ or $\overline{e}$. Thus, we will have found an equilibrium business cycle if c^*_b is again equal to c when e has reached $\overline{e}$ and c^*_s is again equal to c when e has reached $\underline{e}$.

First consider a boom beginning at $t = 0$ with $e(0) = \underline{e}$. Solving (13) for $c^* > c$ yields

$$e_b(t) = -1/2(\alpha + a\ (\alpha-a)\beta \exp(\alpha t))/(1 - \beta \exp(a t))) \qquad (14)$$

where $\alpha = (4a + a^2)^{1/2}$ and $\beta = (2\underline{e} + a + \alpha)/(2\underline{e} + a - \alpha)$. Substituting this into the resulting linear differential equation for c^*_b yields

$$c_b^*(t) = -\mu(t)\ [\int_0^t [\mu(s)]^{-1}(ac + e(s)u)ds - \frac{c}{1-\beta}]\ , \qquad (15)$$

where $\mu(t)$, the integrating factor, is

$$\exp((r + a/2 - \alpha/2)t) - \beta\exp((r + a/2 + \alpha/2)t) \qquad (16)$$

For a slump beginning at time t = 0 with $e(0) = \overline{e}$, we have

$$e_s(t) = \overline{e}/(1+\overline{e}t) \qquad (17)$$

Inserting (19) into (13) and solving yields

$$c^*{}_s(t) = (1 + \overline{e}t)\ \exp(rt)\ [c - \int_0^t \frac{\exp(-rs)\overline{e}uds}{(1 + \overline{e}s)^2} \qquad (18)$$

For a business cycle with floor and ceiling $\underline{e}$ and $\overline{e}$, we follow (14) from the start of the cycle (with $e=\underline{e}$) until the ceiling is hit ($e=\overline{e}$). Then the economy follows (17), until the floor is reached again. Denoting the lengths of boom and slump by t_b and t_s, we have

$$e_b(t_b) = \overline{e}\ ,\ e_s(t_s) = \underline{e} \qquad (19)$$

where e_b ( ) is defined by (14); and e_s( ), by (17).

From equations (17) and (19), we have

$$t_s = 1/\underline{e} - 1\ /\ \overline{e} \qquad (20)$$

Similarly, t_b is uniquely determined from (14) and (19).

For this path to be a rational expectations equilibrium, e_b^* and c_s^* must behave appropriately over the cycle. That is, we need

$$c_b^*(t_b) = c,\ c_b^*(t) \geq c \qquad \text{for } 0 \leq t \leq t_b \qquad (21)$$

$$c_s^*(t_s) = c,\ c_s^*(t) \leq c \qquad \text{for } 0 \leq t \leq t_s$$

where c_b^* and c_s^* are given by (15) and (18).

The search for an economy that has an equilibrium business cycle with floor and ceiling $(\underline{e}, \overline{e})$, with $0 < \underline{e} < e_1 < \overline{e} < e_2$, is the search for three parameters, $a > 0$, $r > 0$, and c/u, with $0 < c/u < 1$ for which (21) holds. (Computationally, we found this an easier question than the more natural question of whether there exists a self-fulfilling floor and ceiling for a given economy.) We claim that if we can satisfy the equality constraints in (21), the inequalities will be satisfied as well. This follows from $\underline{e} < e_1 < \overline{e}$ and inspection of the directions of motion of the system.

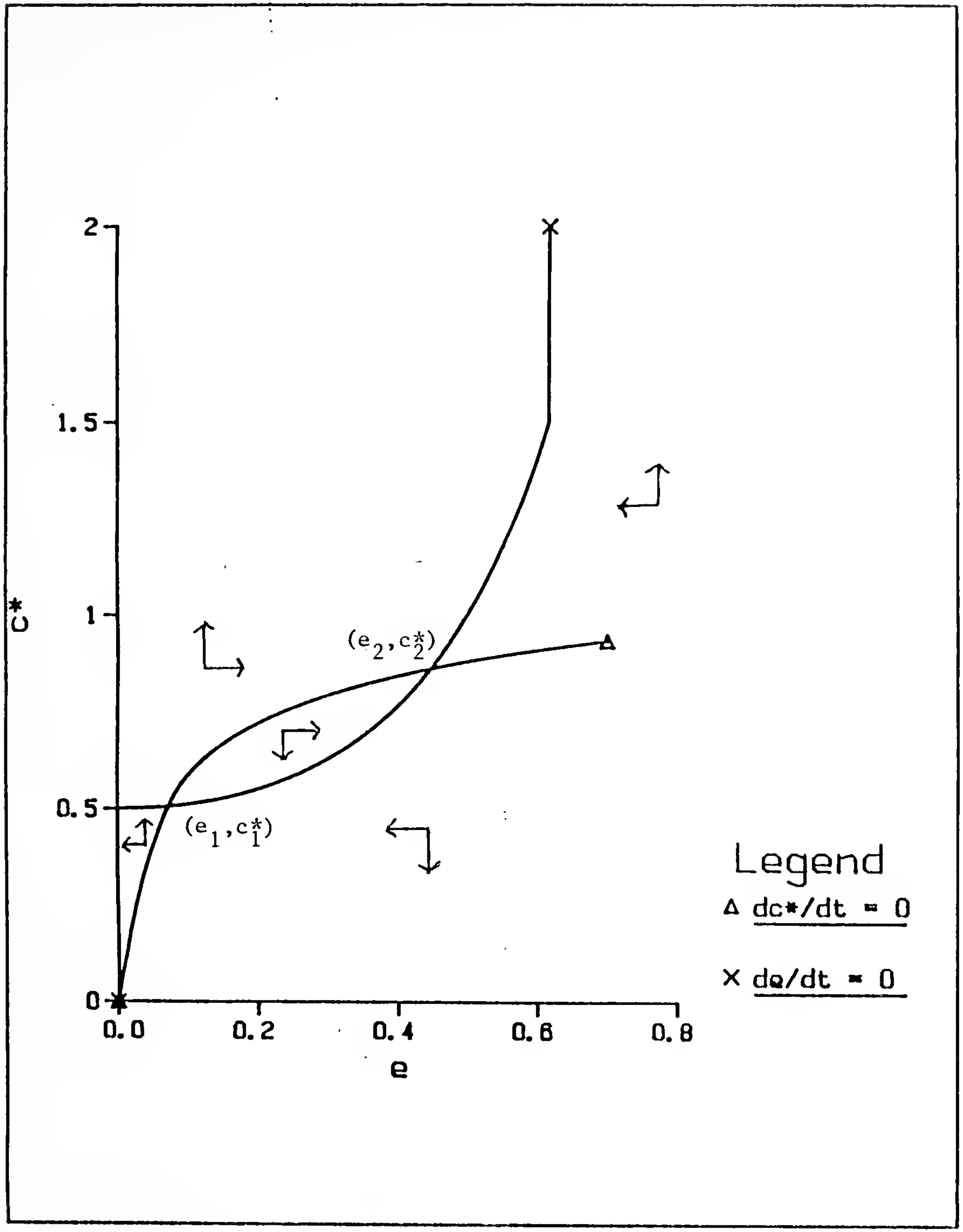

Figure 1

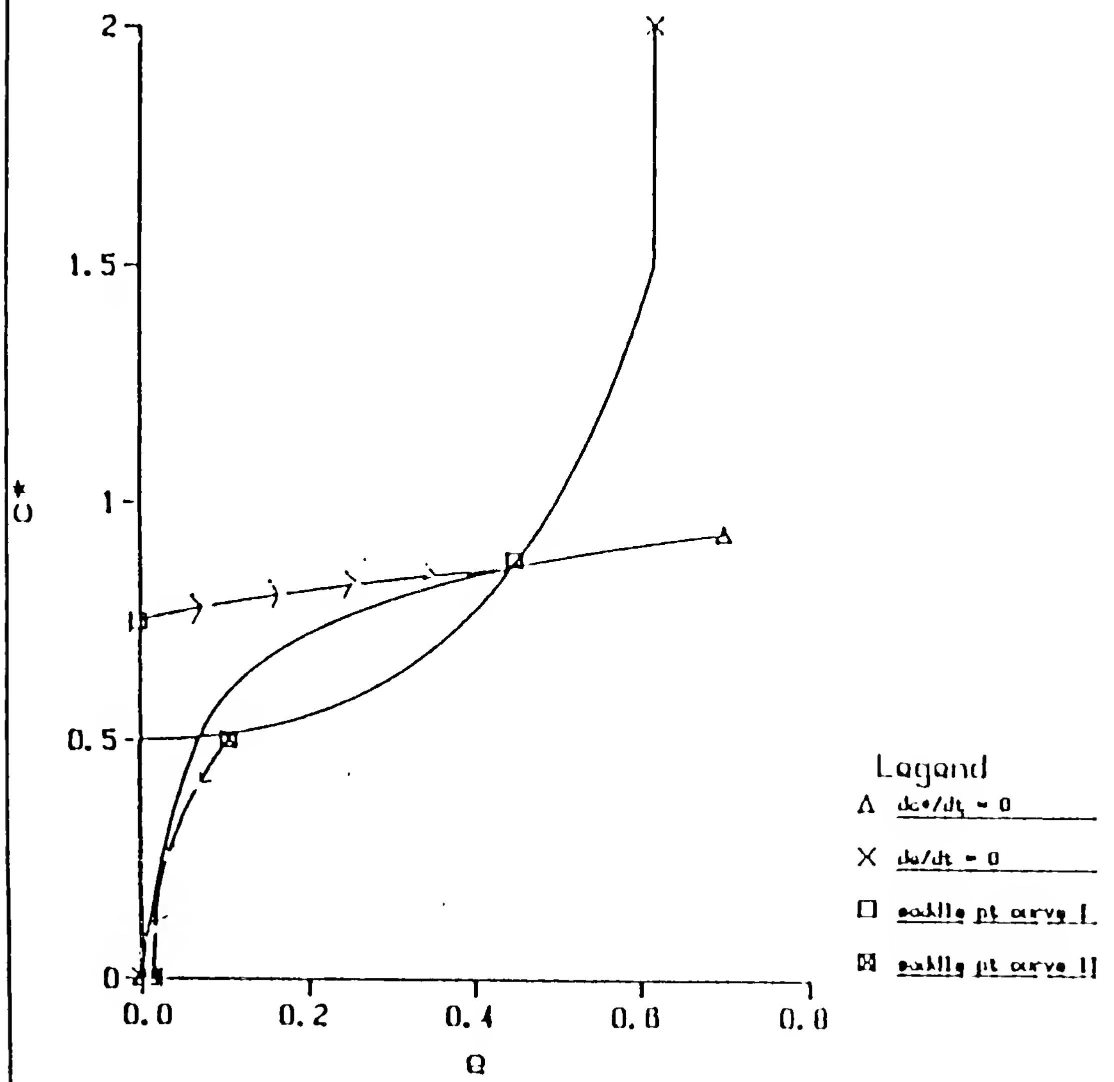

Figure 2

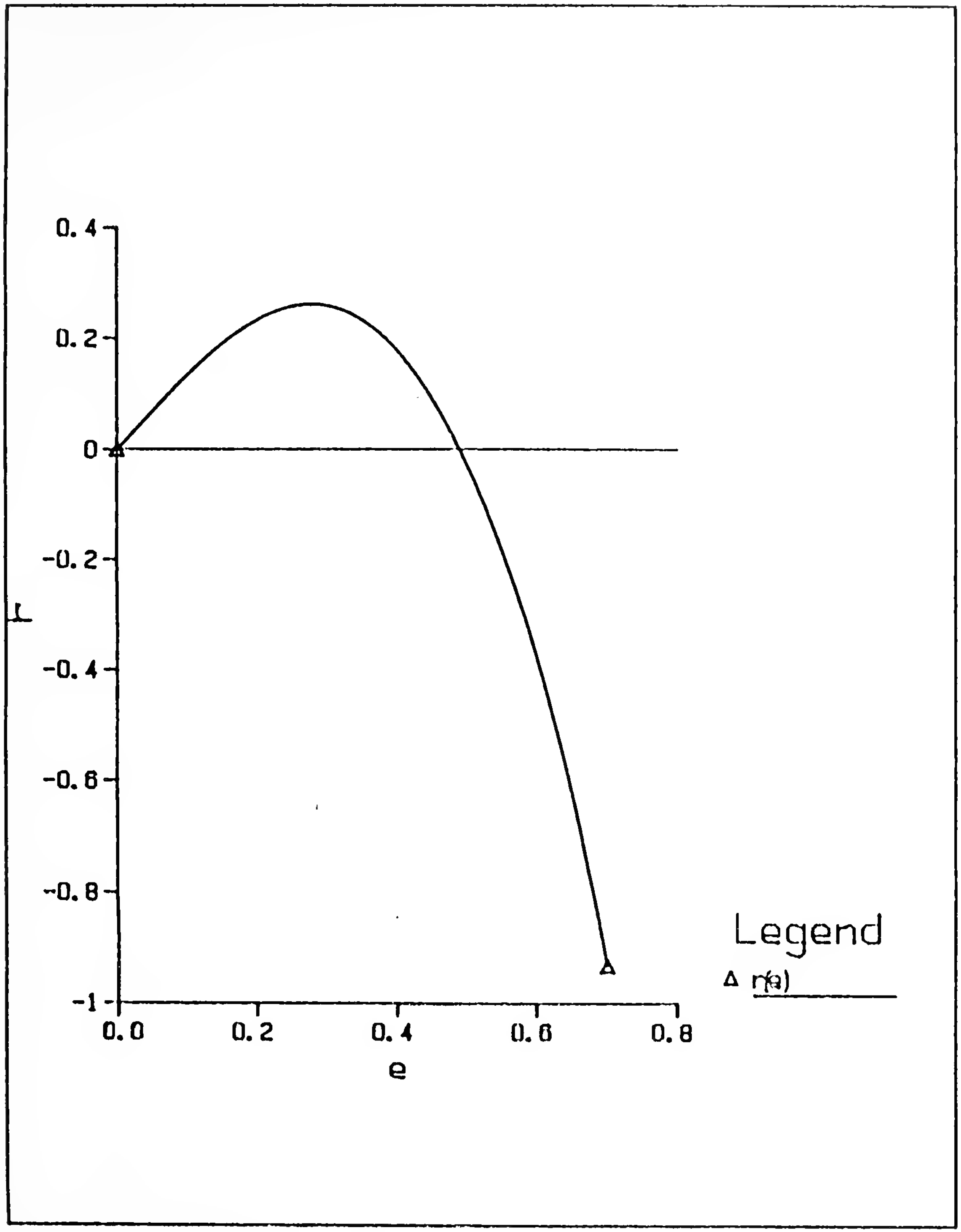

Figure 3

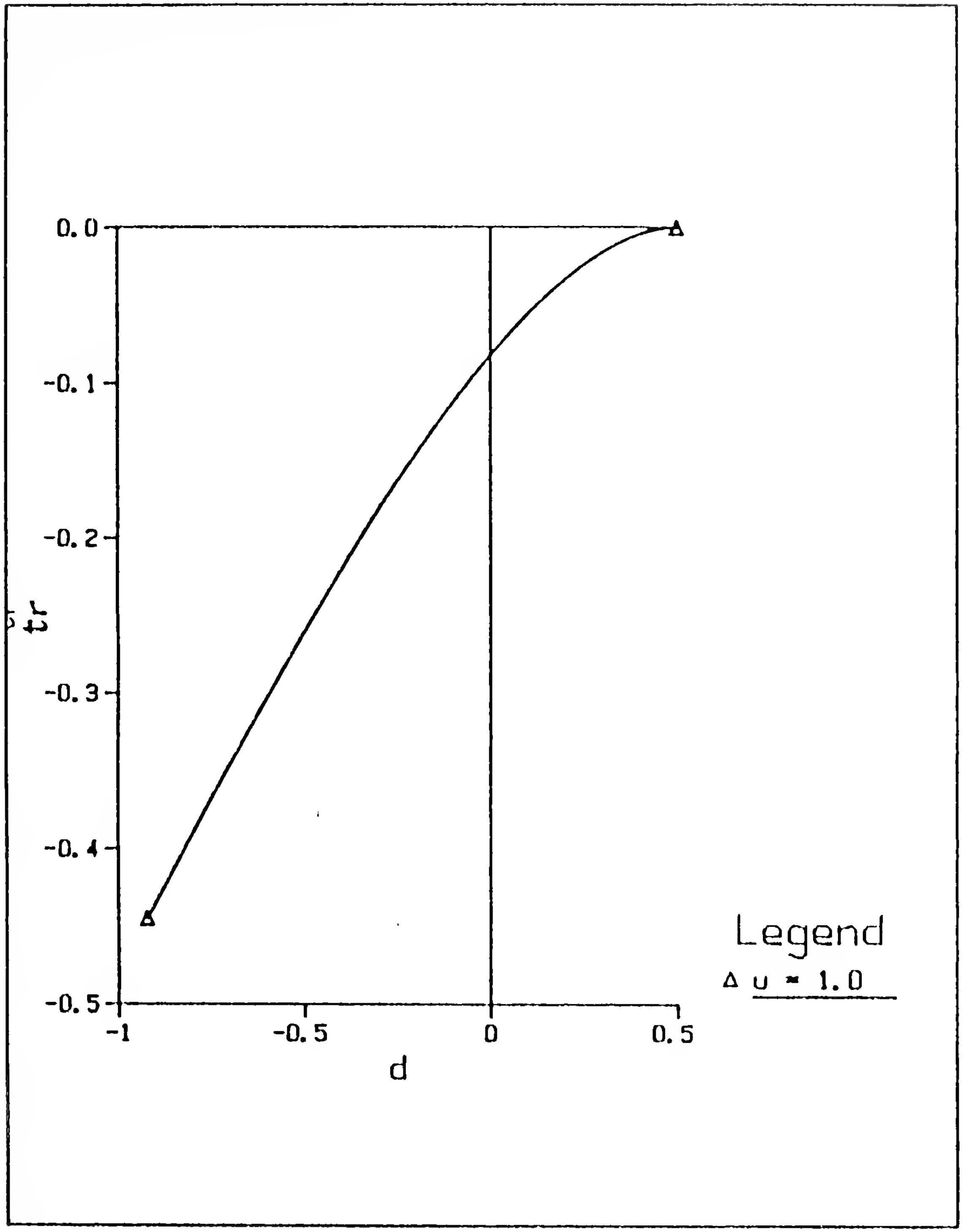

Figure 4a

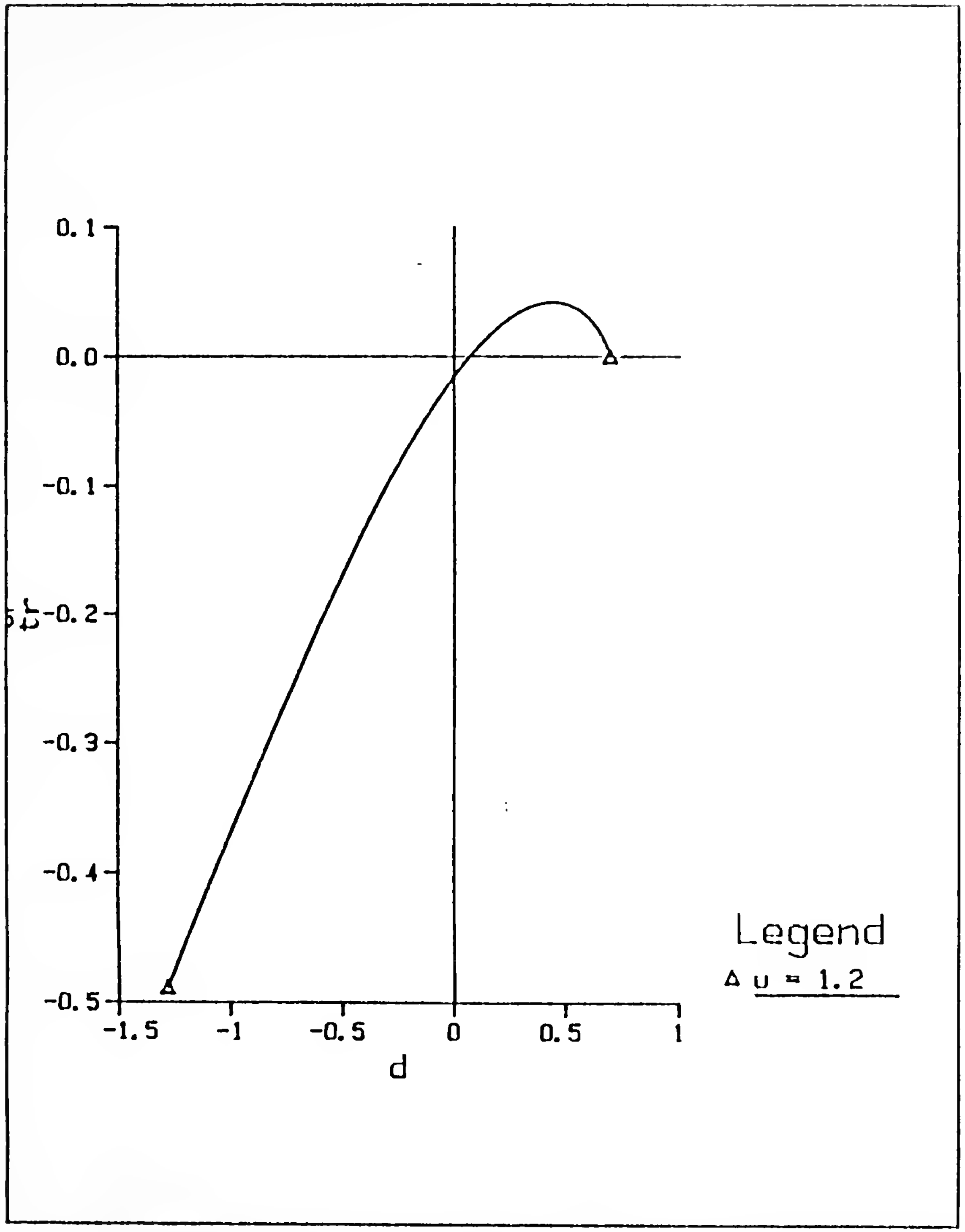

Figure 4b

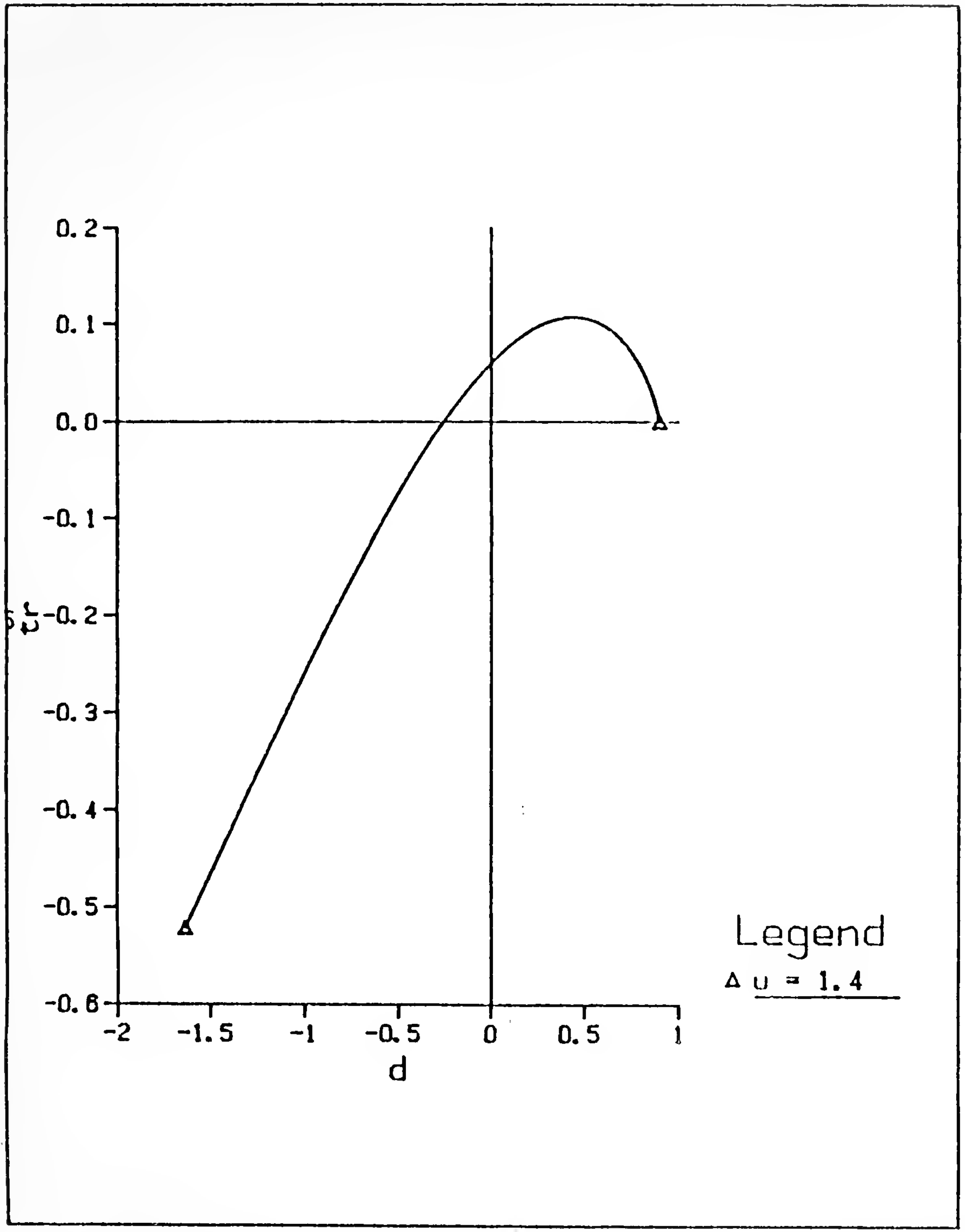

Figure 4c

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